

Symmetry based parameterizations of the lepton mixing matrix

Sanjeev Kumar*

*Department of Physics and Astrophysics, University of Delhi,
Delhi -110007, INDIA.*

(Dated: May 6, 2013)

There are many mixing schemes based upon flavor symmetries that predict a vanishing θ_{13} . These mixing schemes need corrections or modifications to account for recent experimental measurements of non-zero θ_{13} . We propose new parameterizations for the lepton mixing matrix to quantify the minimal modifications needed in these mixing schemes. The parameterizations can be factorized in two parts: $U_0(a, b)$ and $R(\theta, \phi)$. The first factor can be viewed as a zeroth order mixing matrix coming from some flavor symmetry. It reproduces the popular mixing schemes based upon flavor symmetries for suitable values of a and b . The second factor can be interpreted as a minimal modification to the mixing matrix and is responsible for non zero θ_{13} , non-maximal θ_{23} and CP violation. We also find the experimentally allowed parameter space for the parameters a and b and compare it with the symmetry based values for these parameters.

I. INTRODUCTION

The vastly different fermion masses and mixings point to a broken flavor symmetry that might be hidden in the family structure of fermions. In a class of lepton mass models, the residual flavor symmetry in the lepton mass matrices can be related to the original flavor symmetry \mathcal{G} of the Lagrangian [1, 2]. In such models, two different symmetries in the charged lepton and neutrino sector, G_l and G_ν , are preserved when the original symmetry \mathcal{G} is broken. These different symmetries are responsible for different diagonalization matrices for the charge lepton mass matrix M_l and effective neutrino mass matrix M_ν . Thus, a flavor symmetry can predict the neutrino mixing matrix U . As an illustration, bimaximal mixing (BM) [3] and tri-bimaximal mixing (TBM) [4] are based upon the symmetry group S_4 or a larger group containing S_4 as a subgroup [1, 5].

The mixing schemes like BM and TBM are called full mixing schemes since they predict all the three columns of U [1]. Other examples are golden ratio mixing of type I (GRM1) [6] and type II (GRM2) [7], hexagonal mixing (HM) [8] and democratic mixing (DM) [9]. These mixing schemes predict a vanishing θ_{13} and maximal θ_{23} and can at best serve as leading-order approximations to the neutrino mixing matrix. The mass models producing these mixing schemes predict corrections to the mixing angles at next to leading order that are usually of the same magnitude for all angles. This makes it difficult to accommodate a relatively large θ_{13} in the lepton mass models based upon complete mixing that predict zero θ_{13} at the leading order.

Another way of having deviations from a full mixing scheme was suggested by Lam [1, 10] in the form of partial mixing schemes. A partial mixing matrix of type C_i (R_i) is defined as a unitary matrix with i th column (row) fixed to $N\{a \ b \ 1\}^T$ ($N\{1 \ b \ a\}$), while keeping the other

two columns free within unitarity constraints. Here, the parameters a and b are fixed by the flavor symmetry that is responsible for the corresponding complete mixing scheme and $N = 1/\sqrt{1+a^2+b^2}$ is the normalization constant. For example, μ - τ symmetry is a partial mixing of type C_3 with $a = 0$ and $b = 1$ and trimaximal mixing (TM) [11] is a partial mixing of type C_2 with $a = 1$ and $b = 1$. Similarly, one can obtain several partial mixing matrices of the types C_i and R_i from complete mixing schemes like BM, TBM, DM, GRM1, GRM2, DM and HM by selecting respective values of a and b listed in Table 1. We remark that the value of θ_{13} in the partial mixing schemes C_3 and R_1 remains unaltered from its value in the corresponding complete mixing scheme. Hence, these schemes are not of much interest to our work and have not been tabulated in Table 1.

The recent conclusive measurements of a finite θ_{13} [12–15] have initiated an exploration of new leading order approximations to the mixing matrix with non-zero θ_{13} and non-maximal θ_{23} that could result from some larger symmetry groups. Two mixing schemes based upon the modular group were proposed recently by Toorop, Feruglio and Hagedorn [16] which will be called TFH1 and TFH2 mixing schemes. Again, these mixing schemes need next to order corrections to explain three mixing angles simultaneously. One way to do this is by constructing partial mixing schemes for these mixing patterns. We have also included these mixing schemes in Table 1.

The plan of this paper is as follows. In section 2, we review the link between residual symmetry of the neutrino mass matrix and the neutrino mixing matrix. In section 3, we first present six parameterizations for the neutrino mixing matrix with four free parameters. These parameterizations are best suited to describe the partial mixing matrices and can be used to study the corrections in the popular mixing schemes mentioned above. There are many recent studies where similar modifications to various mixing matrices have been studied [18]. Here, we use our parameterizations and obtain sum-rules for mixing angles and CP violation in a model-independent

* skverma@physics.du.ac.in

Mixing pattern	C_1		C_2	
	a	b	a	b
BM	$\sqrt{2}$	1	$\sqrt{2}$	1
TBM	2	1	1	1
DM	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$
GRM1	$\sqrt{3+\sqrt{5}}$	1	$\sqrt{3-\sqrt{5}}$	1
GRM2	$\sqrt{2+\frac{4}{\sqrt{5}}}$	1	$\sqrt{10-4\sqrt{5}}$	1
HM	$\sqrt{6}$	1	$\sqrt{\frac{2}{3}}$	1
TFH1	$\frac{1}{2}(\sqrt{3}+1)$	$\frac{1}{2}(\sqrt{3}-1)$	1	1
TFH2	$2+\sqrt{3}$	$1+\sqrt{3}$	1	1

Mixing pattern	R_2		R_3	
	a	b	a	b
BM	$\sqrt{2}$	1	$\sqrt{2}$	1
TBM	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{2}$
DM	2	1	1	1
GRM1	$\sqrt{\frac{1}{2}(5+\sqrt{5})}$	$\sqrt{\frac{1}{2}(3+\sqrt{5})}$	$\sqrt{\frac{1}{2}(5+\sqrt{5})}$	$\sqrt{\frac{1}{2}(3+\sqrt{5})}$
GRM2	$\sqrt{2+\frac{2}{\sqrt{5}}}$	$\frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{2+\frac{2}{\sqrt{5}}}$	$\frac{1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$
HM	1	$\sqrt{3}$	1	$\sqrt{3}$
TFH1	$2+\sqrt{3}$	$1+\sqrt{3}$	1	1
TFH2	1	1	$2+\sqrt{3}$	$1+\sqrt{3}$

TABLE I. The values of the parameters a and b for the partial mixing schemes matrices of type C_2 , C_3 , R_2 and R_3 generated from several popular complete mixing schemes. The j th column (row) of the mixing matrix of type C_j (R_j) is given by $N\{a \ b \ 1\}^T$ ($N\{1 \ b \ a\}$) and the other two columns (rows) are free within unitarity constraints.

manner. In section 4, we show that these parameterizations can be factorized in two parts: the first part can be identified with the complete mixing schemes and the second part can be thought of a minimal modification or perturbation to these mixing schemes. In section 5, we perform a model independent analysis for the allowed parameter space of these parameterizations in the light of current experimental data and highlight the implications.

II. FLAVOR SYMMETRY AND LEPTON MIXING

If \mathcal{G}_l is the residual symmetry of the charged lepton mass matrix M_l and \mathcal{G}_ν is the residual symmetry of the effective neutrino mass matrix M_ν , M_l and M_ν are invariant under the residual symmetry transformations: $F^\dagger M_l F = M_l$ and $G_i^T M_\nu G_i = M_\nu$ ($i = 1, 2$ and 3). Here, the symmetry transformation F is the diagonal generator of the group $\mathcal{G}_l = Z_n$ ($n \geq 3$) in the diagonal M_l basis and the symmetry transformations G_i 's

$$\begin{aligned}
G_1 &= U \text{diag}(1, -1, -1) U^\dagger \\
G_2 &= U \text{diag}(-1, 1, -1) U^\dagger \\
G_3 &= U \text{diag}(-1, -1, 1) U^\dagger,
\end{aligned} \tag{1}$$

form the group $\mathcal{G}_\nu = Z_2 \times Z_2$ [1, 2]. The residual symmetries, described by the subgroups \mathcal{G}_l and \mathcal{G}_ν , are rem-

nants of the original flavor symmetry \mathcal{G} . When the \mathcal{G} is broken into two different symmetry groups, the charged lepton mass matrix and the neutrino mass matrix may still be invariant under generators of the two residual groups \mathcal{G}_l and \mathcal{G}_ν . In a dynamic model, this is assured if the vacuum alignments of the flavon fields are the invariant eigenvectors of the generators of the symmetry groups of the residual flavor symmetry [1, 2]. To obtain a full mixing matrix, the flavon fields coupling with neutrinos must satisfy three invariance conditions whereas to obtain partial mixing matrix, they have to satisfy only one invariance condition. Hence, the partial mixing is less restrictive than full mixing and ideal for the present experimental scenario.

Several models exist in literature that predict partial mixing schemes of type C_1 [19] and C_2 [20] obtained from TBM mixing. A simple recipe that transforms a TBM model to other mixing schemes like GR is given in Ref. [21]. A general approach based upon group theory for the construction of a dynamic model based upon type I and type II seesaw mechanisms is discussed in Ref. [1]. A generalization of $Z_2 \times Z_2$ symmetry in the neutrino mass matrix can give rise to partial mixings of type C_1 and C_2 [22].

One advantage of such models is that they yield relations between the mixing angles and the CP violating phase instead of completely fixing the mixing an-

gles. Such phenomenological relations have already been studied for special cases such as the trimaximal mixing matrix [11], matrices obtained from $\mu - \tau$ symmetry [23], and a partial mixing scheme obtained from the first column of the tri-bimaximal mixing [19]. However, a general parametrization applicable for all such partial mixing matrices does not exist at present. Consequently, the phenomenological relations resulting from partial mixing have also not been generalized so that they can be studied in a model independent context. Such model-independent relations will be verifiable in future neutrino experiments [12, 24] that are sensitive to the θ_{23} octant and CP violation in neutrino oscillations. Hence, they can be used to distinguish different mixing schemes resulting from the different residual flavor symmetries and, if possible, to reconstruct the original flavor symmetry.

III. THE PARAMETERIZATIONS

The most general mixing matrix with the first column fixed to $N\{a \ b \ 1\}^T$ can be written as

$$U_{(23)} = \begin{pmatrix} aN & N\sqrt{1+b^2}\cos\theta & N\sqrt{1+b^2}\sin\theta \\ bN & \frac{e^{i\phi}\sin\theta - abN\cos\theta}{\sqrt{1+b^2}} & -\frac{e^{i\phi}\cos\theta + abN\sin\theta}{\sqrt{1+b^2}} \\ N & -\frac{aN\cos\theta + be^{i\phi}\sin\theta}{\sqrt{1+b^2}} & \frac{be^{i\phi}\cos\theta - aN\sin\theta}{\sqrt{1+b^2}} \end{pmatrix}. \quad (2)$$

For the mathematical proof, the reader is directed to the appendix A. The other parameterization with second column fixed to $N\{a \ b \ 1\}^T$ is given by

$$U_{(13)} = \begin{pmatrix} N\sqrt{1+b^2}\cos\theta & aN & N\sqrt{1+b^2}\sin\theta \\ \frac{e^{i\phi}\sin\theta - abN\cos\theta}{\sqrt{1+b^2}} & bN & -\frac{e^{i\phi}\cos\theta + abN\sin\theta}{\sqrt{1+b^2}} \\ -\frac{aN\cos\theta + be^{i\phi}\sin\theta}{\sqrt{1+b^2}} & N & \frac{be^{i\phi}\cos\theta - aN\sin\theta}{\sqrt{1+b^2}} \end{pmatrix}. \quad (3)$$

A nice property of these parameterizations is that the (1,3) element vanishes in the special case $\theta = 0$. We do not consider the parameterization $U_{(12)}$ because its third column will be $N\{a \ b \ 1\}^T$ and, therefore, the (1,3) element does not vanish for $\theta = 0$.

Similarly, the parameterizations with the second and third rows being equal to $N\{1 \ b \ a\}$ are given by

$$U^{(13)} = \begin{pmatrix} \frac{be^{i\phi}\cos\theta - aN\sin\theta}{\sqrt{1+b^2}} & -\frac{e^{i\phi}\cos\theta + abN\sin\theta}{\sqrt{1+b^2}} & N\sqrt{1+b^2}\sin\theta \\ N & bN & aN \\ -\frac{aN\cos\theta + be^{i\phi}\sin\theta}{\sqrt{1+b^2}} & \frac{e^{i\phi}\sin\theta - abN\cos\theta}{\sqrt{1+b^2}} & N\sqrt{1+b^2}\cos\theta \end{pmatrix} \quad (4)$$

and

$$U^{(12)} = \begin{pmatrix} \frac{be^{i\phi}\cos\theta - aN\sin\theta}{\sqrt{1+b^2}} & -\frac{e^{i\phi}\cos\theta + abN\sin\theta}{\sqrt{1+b^2}} & N\sqrt{1+b^2}\sin\theta \\ -\frac{aN\cos\theta + be^{i\phi}\sin\theta}{\sqrt{1+b^2}} & \frac{e^{i\phi}\sin\theta - abN\cos\theta}{\sqrt{1+b^2}} & N\sqrt{1+b^2}\cos\theta \\ N & bN & aN \end{pmatrix}, \quad (5)$$

receptively. Again, the (1,3) element of the mixing matrix vanishes for $\theta = 0$ in these parameterizations. The parameterization $U^{(23)}$ with first row equal to $N\{1 \ b \ a\}$ will not share this property and, hence, are not studied here. We further note that many other parameterizations of similar nature can be constructed by appropriate permutations of the elements of the above mixing matrices. The choices for the positions of the elements we have made are not unique but motivated by simplicity of results.

If we substitute the values of a and b in the parameterizations $U_{(23)}$, $U_{(13)}$, $U^{(13)}$ and $U^{(12)}$ that are listed in Table 1, we will get the corresponding partial mixing schemes of types C_1 , C_2 , R_2 and R_3 , respectively. Further, if we put $\theta = 0$ and $\phi = 0$ or π , we get the complete mixing schemes listed in Table 1 for the respective values of a and b . So, these parameterizations are ideal for

studying any partial mixing scheme in a model independent way. We also note that the parameters a and b are to be determined from the experimental data whereas the values listed in Table 1 may be viewed as (zeroth order) predictions of some flavor symmetries. The comparison between the experimentally allowed values of a and b with the “symmetry-based values” given in Table 1 form the bases of our model independent analysis.

Just like all other unitary parameterizations for the mixing matrix, the above parameterizations have four free parameters a , b , θ and ϕ . The mixing angles θ_{12} , θ_{23} and θ_{13} and CP violating phase δ in the PDG parameterization can be expressed in terms of these four parameters. The expressions for the mixing angle θ_{13} and the Jarlskog rephasing invariant measure of CP violation, $J = \text{Im}(U_{11}U_{12}^*U_{21}^*U_{22})$ [17], are identical for all

Parameterization	$\sin^2 \theta_{12}$	$\sin^2 \theta_{23}$
$U_{(23)}$	$1 - \frac{a^2}{a^2 + (1+b^2) \cos^2 \theta}$	$\sin^2 \theta_0 + A \cos \phi$
$U_{(13)}$	$\frac{a^2}{a^2 + (1+b^2) \cos^2 \theta}$	$\sin^2 \theta_0 + A \cos \phi$
$U^{(13)}$	$\sin^2 \theta_0 + A \cos \phi$	$\frac{a^2}{a^2 + (1+b^2) \cos^2 \theta}$
$U^{(13)}$	$\sin^2 \theta_0 + A \cos \phi$	$1 - \frac{a^2}{a^2 + (1+b^2) \cos^2 \theta}$

TABLE II. The expressions for θ_{12} and θ_{23} in terms of the parameterizations proposed in the present work.

the four parameterizations:

$$\sin^2 \theta_{13} = N^2 (1 + b^2) \sin^2 \theta \quad (6)$$

and

$$J = B \sin \phi \quad (7)$$

where

$$B = abcN^3 \sin \theta \cos \theta. \quad (8)$$

The expressions for the other two mixing angles are different in different parameterizations and have been listed in Table 2. The parameters θ_0 and A used in Table 2 are given by

$$\sin^2 \theta_0 = \frac{1}{1 + b^2} \left(1 - \frac{a^2(1 - b^2) \sin^2 \theta}{a^2 + (1 + b^2) \cos^2 \theta} \right) \quad (9)$$

and

$$A = \frac{ab \sin 2\theta \sqrt{1 + a^2 + b^2}}{(1 + b^2)(a^2 + (1 + b^2) \cos^2 \theta)}. \quad (10)$$

Using various partial mixing schemes, these relations can be used to obtain sum-rules for the mixing angles and CP violation by eliminating the parameters θ and ϕ . The sum-rules relating θ_{12} and θ_{13} are $\cos \theta_{13} \cos \theta_{12} = aN$ and $\cos \theta_{13} \sin \theta_{12} = aN$ for the partial mixing schemes C_1 and C_2 , respectively. The sum-rule for θ_{23} and J is

$$\frac{(\sin^2 \theta_{23} - \sin^2 \theta_0)^2}{A^2} + \frac{J^2}{B^2} = 1 \quad (11)$$

for both the partial mixing schemes C_1 and C_2 . Similarly, one can easily see that $\cos \theta_{13} \sin \theta_{23} = aN$ and $\cos \theta_{13} \cos \theta_{23} = aN$ for the partial mixing schemes R_2 and R_3 , respectively. For both of these mixing schemes, θ_{12} and J are related as

$$\frac{(\sin^2 \theta_{12} - \sin^2 \theta_0)^2}{A^2} + \frac{J^2}{B^2} = 1. \quad (12)$$

Hence, the relation between θ_{23} (θ_{12}) and J is equation for an ellipse centered around the point $(\sin^2 \theta_0, 0)$ for the mixing schemes C_1 and C_2 (R_1 and R_2). The semi-minor axis of the ellipse equals B , the maximal CP violation J ,

whereas the semi-major axis is given by A , the deviation of $\sin^2 \theta_{23}$ ($\sin^2 \theta_{12}$) from $\sin^2 \theta_0$. A generic prediction of these relations is that CP violation will be maximal for a maximal θ_{23} (θ_{12}) for the mixing schemes C_1 and C_2 (R_1 and R_2).

The detailed phenomenology of the resulting partial mixing schemes have been studied extensively in the literature [18]. The results of some of these studies can readily be obtained from the above relations and sum-rules by substituting respective values of the symmetry parameters a and b listed in Table 1. (However, we differ from many of these studies in the way we introduce the phase ϕ in the mixing matrix.) Since all these studies presume certain values for the parameters a and b at the outset, they do not address the following two questions:

1. What is the allowed parameter space for the parameters a and b ?
2. How the symmetry based values for the parameters a and b (Table 1) compare with the experimentally allowed values for these parameters?
3. Which of the partial mixing matrices are preferred by the current neutrino mixing data?

Here, we shall address these questions in a model independent manner as an important application of the parameterizations we have presented here.

IV. THE FACTORIZATION OF THE MIXING MATRIX

One of the main advantages of the parameterizations proposed here is that the mixing matrix can be factorized in two parts:

$$U_{(ij)} = V_{(ij)}(a, b) R_{(ij)}(\theta, \phi) \quad (13)$$

and

$$U^{(ij)} = R^{(ij)}(\theta, \phi) V^{(ij)}(a, b). \quad (14)$$

In these equations, the matrices $V_{(ij)}(a, b)$ and $V^{(ij)}(a, b)$ are simply the value of $U_{(ij)}$ and $U^{(ij)}$ for $\theta = 0$ and $\phi = 0$. The complex rotations are given by $R_{(ij)}(\theta, \phi) = P(\phi) O_{(ij)}(\theta)$ and $R^{(ij)}(\theta, \phi) = O_{(ij)}(\theta) Q(\phi)$. Here, the indices $\{i, j, k\}$ are cyclic permutations of $\{1, 2, 3\}$. The matrix $O_{(ij)}$ is an orthogonal (i, j) rotation by angle θ , $P(\phi) = \text{diag}\{1, 1, e^{i\phi}\}$ and $Q(\phi) = \text{diag}\{e^{i\phi}, 1, 1\}$.

The factorization of the mixing matrix in the two parts $V(a, b)$ and $R(\theta, \phi)$ has many important implications. One can associate the $V(a, b)$ part with a complete mixing scheme like TBM and the $R(\theta, \phi)$ part with a modification or perturbation to that scheme. We note that the $V(a, b)$ part reduces to the complete mixing schemes listed in Table 1 for the respective values of the parameters a and b (except for TFH1 and TFH2 where θ_{13} is

non-zero in the complete mixing scheme itself). Therefore, the perturbation $R(\theta, \phi)$ affects only two parameters (θ and ϕ) from their values in the corresponding full mixing ($\theta = 0$ and $\phi = 0$). The parameters a and b remain unaffected by the perturbation. In other words, the parametrizations proposed here parametrize not only the four experimental observables in the mixing matrix in terms of the four parameters a , b , θ and ϕ , they can also parameterize the neutrino mass matrices giving rise to partial mixing schemes. In this model building context, the $V(a, b)$ part can come from a residual flavor symmetry and the $R(\theta, \phi)$ part can result from some symmetry breaking terms.

The parametrizations $U^{(ij)}$ have another interesting property. The lepton mixing matrix can be written as $V_l^\dagger V_\nu$, where V_l and V_ν are the unitary matrices that diagonalize the charged lepton and effective neutrino mass matrices, respectively. A comparison with Eq. (14) yields $V_l = R^{(ij)\dagger}(\theta, \phi) = Q(-\phi)O_{(ij)}^T(\theta)$ and $V_\nu = V^{(ij)}(a, b)$.

V. ANALYSIS AND RESULTS

We first perform a Monte-Carlo analysis of the allowed parameter space for the parameters a and b to get the preferences in the present experimental data for various partial mixing schemes. We start with generating n random samples for the parameters a , b , θ and ϕ with uniform distributions in the ranges $\{a_{min}, a_{max}\}$, $\{b_{min}, b_{max}\}$, $\{\theta_{min}, \theta_{max}\}$ and $\{\phi_{min}, \phi_{max}\}$. These values are used to calculate the mixing angles θ_{12} , θ_{23} and θ_{13} from the relations given in Section 3. The allowed parameter space on the plane (a, b) is depicted in Fig. 1 where the three mixing angles θ_{12} , θ_{23} and θ_{13} are consistent with their experimental values [26] at three standard deviations for all the four parameterizations. The ranges $\{a_{min}, a_{max}\}$, $\{b_{min}, b_{max}\}$, $\{\theta_{min}, \theta_{max}\}$ and $\{\phi_{min}, \phi_{max}\}$ are decided by the following algorithm: begin with arbitrarily small ranges, say $\{0, 1\}$, and then go on gradually expanding them till the allowed parameter space does not expand further.

The main motivation of the above analysis is to obtain an understanding of the experimental viability of different modifications of some popular mixing schemes. However, it does not give us the best fit values and their experimental errors for the various parameters. This is accomplished by doing a χ^2 analysis for the three mixing angles θ_{12} , θ_{23} and θ_{13} . The allowed regions are depicted in Fig. 1 as contours at 1σ and 2σ C.L. The allowed ranges for the parameters are tabulated in Table 3.

Finally, we superimpose the values of the parameters a and b listed in Table 1 corresponding to various mixing schemes on the allowed parameter regions depicted in Fig. 1. A comparison between the symmetry based values for the parameters a and b (Table 1) and the experimentally allowed values for these parameters can

	a	b	θ
$U_{(23)}$	1.75-3.55	0.73-3.53	0.26-0.29
	1.69-3.89	0.68-2.39	0.24-0.31
	1.64-4.14	0.60-2.53	0.21-0.35
$U_{(13)}$	0.89-1.27	0.91-1.61	0.18-0.20
	0.85-1.36	0.89-1.72	0.17-0.21
	0.67-1.57	0.54-1.88	0.15-0.23
$U^{(13)}$	1.16-1.93	1.11-2.10	0.19-0.21
	1.10-2.07	1.09-2.22	0.18-0.22
	0.99-4.01	0.85-3.01	0.16-0.29
$U^{(12)}$	1.63-3.34	0.96-2.51	0.23-0.26
	1.55-3.69	0.92-2.69	0.22-0.27
	1.07-4.47	0.82-3.12	0.16-0.31

TABLE III. The experimentally allowed values of the parameters a , b and θ for the four parameterizations. The successive rows give the allowed ranges at 1, 2 and 3 σ C.L. The phase ϕ can take any value in its full range $\{0, 2\pi\}$.

be made visually from Fig. 1. The confidence levels at which the various partial mixing schemes are allowed or ruled out by this model-independent analysis are given in Table 4. We note that the partial mixing schemes constructed from the all four partial mixing schemes constructed from DM are disallowed by the current mixing data at more than 3σ C.L. The partial mixing matrices for BM mixing are disallowed at more than 3σ for C_1 and C_2 and at more than 2σ C.L. for R_1 and R_2 . The partial mixing schemes for HM of the types C_2 , R_2 and R_3 are disallowed at 2σ C.L. whereas the partial mixing of type C_1 for HM is disallowed at more than 3σ C.L. In fact, of the partial mixing patterns of type R_2 and R_3 are ruled out at more than 2σ C.L. because of a preference for $\theta_{23} < \pi/4$ in the present mixing data at 2σ C.L. [26]. Most successful mixing pattern is C_1 for TBM. This partial mixing matrix is even more successful than TM mixing (partial mixing of type C_2 for TBM) and have already been studied in the literature [19] where its model realization has also been discussed. Two similar partial mixing matrices of type C_1 for GRM2 and of type C_2 for GR1 are also viable and should be considered for model building.

VI. CONCLUSIONS

In summary, we have presented six parametrizations for the lepton mixing matrix. These parametrizations are useful to describe neutrino mixing and CP violation in any lepton mass model possessing a residual symmetry. The parameterizations are ideal to describe the partial mixing schemes that are minimal modifications of the complete mixing schemes. As an application of these pa-

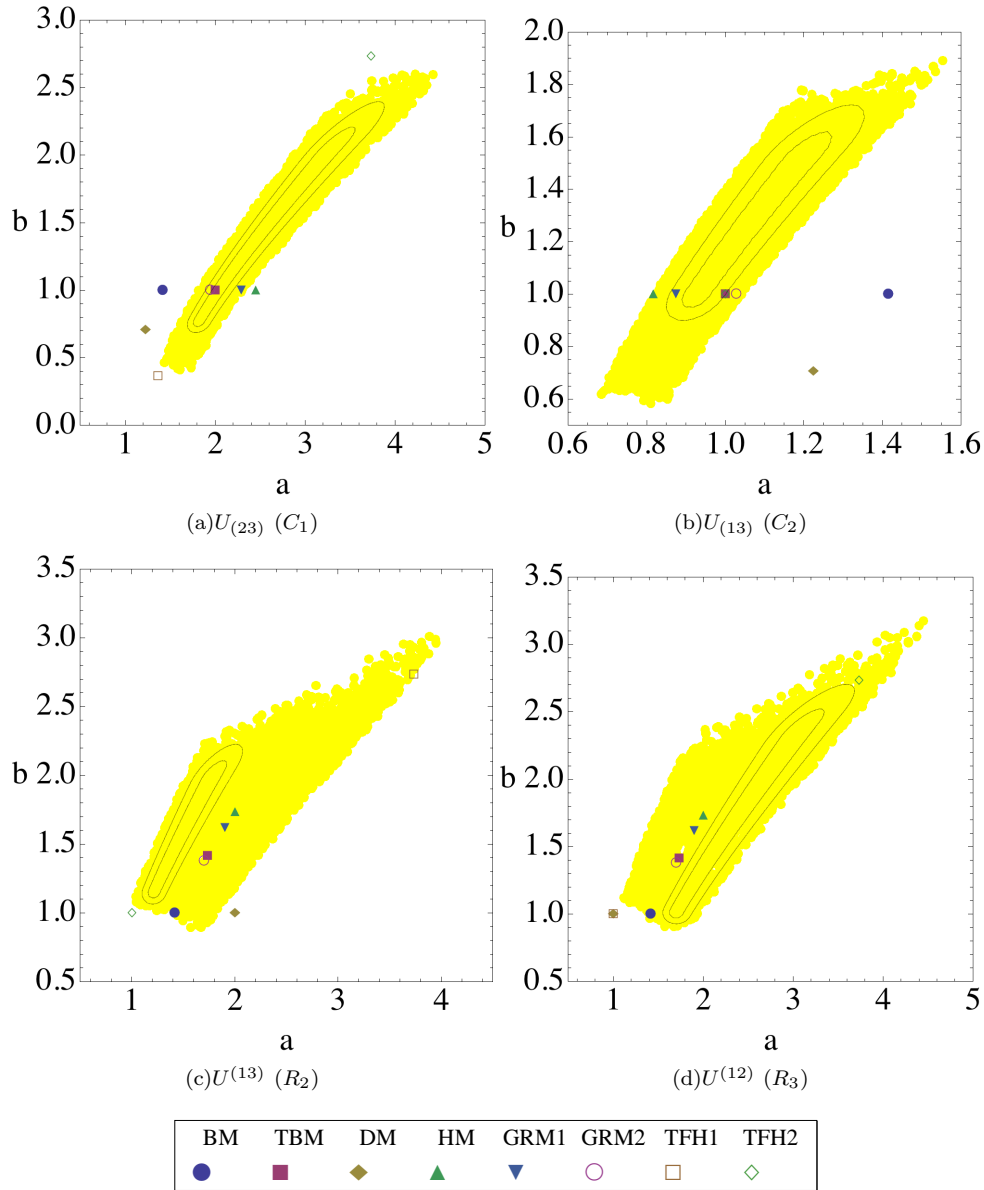


FIG. 1. (color online). The allowed parameter space for the parameters a and b . The two contours give the 1σ and 2σ allowed region for the parameters a and b when the other two parameters (θ and ϕ) are being marginalized away. The gray (yellow) regions depict the allowed parameter space at 3σ . The symmetry values of a and b for corresponding to various mixing patterns (Table 1) are also depicted for comparison. The values of a and b coincide for TBM, TFH1 and TFH2 for C_2 and TFH1 and DM for R_3 .

parameterizations, we obtain interesting sum-rules for neutrino mixing angles and CP violation for the partial mixing schemes in a model independent manner. These parameterizations can be factored into two parts: $U_0(a, b)$ and $R(\theta, \phi)$. The $U_0(a, b)$ part can be considered as the zeroth order prediction of a flavor symmetry and the $R(\theta, \phi)$ part can be considered as a correction to it. We find the experimentally allowed parameter space for the parameters a and b . The allowed ranges of these parameters can be interpreted as the model-independent predictions for a neutrino mass matrix with Z_2 symmetry. We compare

these predictions for the parameters a and b with their values in different partial mixing schemes. This model independent analysis favors the minimal modifications of TBM and GRM schemes over the modifications of BM, DM and HM mixing.

It is a pleasure to thank Samrajit Triambak and Radha Raman Gautam for critical reading of the manuscript and helpful suggestions. This work is supported by the Department of Science and Technology (DST), Government of India *vide* grant number SR/FTP/PS-123/2011.

	C_1	C_2	R_2	R_3
BM	$> 3\sigma$	$> 3\sigma$	$> 2\sigma$	$> 2\sigma$
TBM	$< 1\sigma$	$< 2\sigma$	$> 2\sigma$	$> 2\sigma$
DM	$> 3\sigma$	$> 3\sigma$	$> 3\sigma$	$> 3\sigma$
HM	$> 3\sigma$	$> 2\sigma$	$> 2\sigma$	$> 2\sigma$
GRM1	$> 2\sigma$	$< 2\sigma$	$> 2\sigma$	$> 2\sigma$
GRM2	$< 2\sigma$	$> 2\sigma$	$> 2\sigma$	$> 2\sigma$
TFH1	$> 3\sigma$	$< 2\sigma$	$> 2\sigma$	$> 3\sigma$
TFH2	$> 3\sigma$	$< 2\sigma$	$> 3\sigma$	$> 2\sigma$

TABLE IV. The confidence levels by which the various partial mixing schemes listed in Table 1 are allowed or disallowed by the neutrino mixing data. The symbol $> n\sigma$ means that the corresponding mixing scheme is disallowed at more than $n\sigma$ C.L. The symbol $< n\sigma$ means that the corresponding mixing matrix is allowed and the disagreement with the experimental data is less than $n\sigma$ C.L.

Appendix A: The parameterization $U_{(23)}$

As an illustration, we will find a mixing matrix of type

$$U = \begin{pmatrix} aN & u_1 & v_1 \\ bN & u_2 & v_2 \\ N & u_3 & v_3 \end{pmatrix} \quad (\text{A.1})$$

from the unitarity constraints. Here, $u_1 = x_1 + iy_1$, $v_1 = x_2 + iy_2$, $u_2 = x_3 + iy_3$ and $v_2 = x_4 + iy_4$. The orthogonalization of the columns of U yields $u_3 = -(au_1 + bu_2)$ and $v_3 = -(av_1 + bv_2)$. Substituting these values in Eq. (A.1), we obtain

$$U = \begin{pmatrix} aN & x_1 + iy_1 & x_2 + iy_2 \\ bN & x_3 + iy_3 & x_4 + iy_4 \\ N & -(ax_1 + bx_3) - i(ay_1 + by_3) & -(ax_2 + bx_4) - i(ay_2 + by_4) \end{pmatrix}. \quad (\text{A.2})$$

Further, solving the unitarity relations $UU^\dagger = U^\dagger U = 1$, we get

$$x_2^2 = 1 - a^2 N^2 - x_1^2 - y_1^2 - y_2^2, \quad (\text{A.3})$$

$$y_3 = \frac{1}{(1+b^2)^2} \{-ab(1+b^2)y_1 + d\}, \quad (\text{A.4})$$

$$x_4 = \frac{abx_2}{1+b^2} - \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \left(x_3 + \frac{abx_1}{1+b^2} \right) - \frac{d(x_1y_2 + y_1x_2)}{(1+b^2)^2(x_2^2 + y_2^2)} \quad (\text{A.5})$$

and

$$y_4 = -\frac{aby_2}{1+b^2} - \frac{x_1y_2 - y_1x_2}{x_2^2 + y_2^2} \left(x_3 + \frac{abx_1}{1+b^2} \right) - \frac{d(x_1x_2 + y_1y_2)}{(1+b^2)^2(x_2^2 + y_2^2)}, \quad (\text{A.6})$$

where

$$d^2 = (1+b^2)^2 [(1+a^2+b^2)c^2(x_2^2 + y_2^2) - \{abx_1 + (1+b^2)x_3\}^2]. \quad (\text{A.7})$$

The parameters y_1 and y_2 give rise to the Majorana phases which can be factored out into unconstrained neutrino masses. Substituting $y_1 = y_2 = 0$, we are effectively left with two free parameters in the mixing matrix *viz.* x_1 and x_3 , a fact that can be checked from simple parameter counting. These parameters can be further reparametrized in terms of two angles θ and ϕ as $x_1 = N\sqrt{1+b^2}\cos\theta$ and $\sqrt{1+b^2}x_3 = \sin\theta\cos\phi - abN\cos\theta$.

With these redefinitions, the most general mixing matrix of type C_1 given by Eq. (A.2) becomes

$$U = \begin{pmatrix} aN & N\sqrt{1+b^2}\cos\theta & N\sqrt{1+b^2}\sin\theta \\ bN & \frac{e^{i\phi}\sin\theta - abN\cos\theta}{\sqrt{1+b^2}} & \frac{-ce^{i\phi}\cos\theta - abN\sin\theta}{\sqrt{1+b^2}} \\ N & \frac{-aN\cos\theta - be^{i\phi}\sin\theta}{\sqrt{1+b^2}} & \frac{be^{i\phi}\cos\theta - aN\sin\theta}{\sqrt{1+b^2}} \end{pmatrix}. \quad (\text{A.8})$$

- (2009).
- [3] V. D. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, Phys. Lett. **B 437**, 107 (1998); G. Altarelli and F. Feruglio, JHEP **11**, 021 (1998); R. N. Mohapatra and S. Nussinov, Phys. Rev. **D 60**, 013002 (1999).
 - [4] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. **B 530**, 167 (2002).
 - [5] C. S. Lam, Phys. Rev. Lett. **101**, 121602 (2008).
 - [6] Y. Kajiyama, M. Raidal, A. Strumia, Phys. Rev. **D 76**, 117301 (2007).
 - [7] W. Rodejohann, Phys. Lett. **B 671**, 267, (2009).
 - [8] C. Giunti, Nucl. Phys. **B**, Proc. Suppl. 117, 24 (2003); Z. -Z. Xing, J. Phys. G 29, 2227 (2003); I. de Medeiros Varzielas, R. Gonzalez Felipe and and H. Serodio, Phys. Rev. **D 83** 033007 (2011).
 - [9] H. Fritzsch, Z. Z. Xing, Phys. Lett. B372 (1996) 265.
 - [10] C. S. Lam, Phys. Rev. **D 74**, 113004 (2006).
 - [11] James D. Bjorken, P. F. Harrison, and W. G. Scott, Phys. Rev. **D 74**, 073012 (2006); C. S. Lam, Phys. Lett. B 656, 193 (2007); Sanjeev Kumar, Phys. Rev. **D 82**, 013010 (2010).
 - [12] K. Abe *et al.* (T2K Collaboration), Phys. Rev. Lett. **107**, 041801 (2011).
 - [13] P. Adamson *et al.* (MINOS Collaboration), Phys. Rev. Lett. **107**, 181802 (2011).
 - [14] F. P. An *et al.* (Daya Bay Collaboration), Phys. Rev. Lett. **108** 171803 (2012).
 - [15] J. K. Ahn *et al.* (RENO Collaboration), Phys. Rev. Lett. **108**, 191802 (2012).
 - [16] Reinier de Adelhart Toorop, Ferruccio Feruglio, Claudia Hagedorn, Nucl. Phys. **B 858** 437 (2012); Stephen F. King, Christoph Luhn, Alexander J. Stuart, Nucl. Phys. **B 867**, 203 (2013).
 - [17] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).
 - [18] Xiao-Gang He and A. Zee, Phys Rev D 84, 053004; Wei Chao and Ya-juan Zheng, JHEP **1302** 044 (2013); Bin Wang, Jian Tang, Xue-Qian Li, arXiv:1303.1592.
 - [19] Stefan Antusch, Stephen F. King, Christoph Luhn and Martin Spinrath, Nuclear Physics B 856 (2012) 328341; Werner Rodejohann and He Zhang, Phys. Rev. **D 86**, 093008 (2012).
 - [20] Iain K. Cooper, Stephen F. King and Christoph Luhn, JHEP **06** 130 (2012); Christoph Luhn, Krishna Mohan Parattu and Akn Wingerter, JHEP **1212** 096 (2012); W. Grimus and L. Lavoura, JHEP **09** 106, (2008); S. F. King and C. Luhn, J. High Energy Phys. **09** 042, (2011); Stefan Antusch, Stephen F. King, Christoph Luhn and Martin Spinrath, Nucl. Phys. **B 856** 328, (2012).
 - [21] I. de Medeiros Varzielas, R. Gonzalez Felipe and H. Serodio, Phys. Rev. **D 83**, 033007 (2011)]
 - [22] Shivani Gupta, Anjan S. Joshipura and Ketan M. Patel, Phys. Rev. **D 85**, 031903(R) (2012).
 - [23] Shao-Feng Ge, Duane A. Dicus and Wayne W. Repko, Phys. Rev. Lett. **108**, 041801 (2012).
 - [24] Gavin S. Davies for the NO ν A Collaboration, Proceedings of the DPF-2011 Conference, 2011, arXiv:1110.0112.
 - [25] M Tortola, J W F Valle and D. Vanegas, Phys. Rev. **D 86** 073012 (2012).
 - [26] G.L. Fogli *et al.*, Phys. Rev. **D 86** 013012 (2012).